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New convergence theorems for Newton-like-iterative methods

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Abstract: Newton-like-iterative methods proposed by T.J. Ypma are obtained by using an iterative method to solve Newton-like equations. In the early paper of Ypma, the theory of inexact Newton methods was applied to study the convergence of Newton-like-iterative methods. Unlike earlier results, new local convergence theorems for Newton-like-iterative methods by applying the theory of inexact Newton-like methods is proposed in this paper, which is seemed simpler and clearer. Moreover, the analysis is carried out in affine invariant terms.

Key words: nonlinear equations; Newton-like methods; Newton-like-iterative methods; inexact Newton method; inexact Newton-like methods; affine invariant

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关于 Newton-like-iterative 方法新的收敛性定理

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摘　要: 用迭代法求解 Newton-like 法中的方程,T.J. Ypma 提出 Newton-like-iterative 方法。在其早期的文章中,不精确牛顿法理论用来研究 Newton-like-iterative 方法的收敛性。与以往方法不同,今提出用不精确 Newton-like 法做相关的收敛性分析,所得定理更加简单,同时具有仿射不变性。

关键词: 非线性方程;Newton-like 方法;Newton-like-iterative 方法;不精确牛顿法;不精确 Newton-like 方法;仿射不变性

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1 Introduction

Newton-like methods for solving the system of nonlinear equations

$$f(x)=0 \quad f: D \subseteq R^N \rightarrow R^N \tag{1}$$

have the general form

$$x_{k+1} = x_k + s_k \quad k=0,1,\cdots, \tag{2}$$

where $s_k \in R^N$ is the solution s of the system of linear equations

$$A_k s = -f(x_k) \quad k=0,1,\cdots, \tag{3}$$

where $A_k \in L(R^N)$ is some nonsingular approximation to the Fréchet derivative $f'(x_k)$ of f at x_k.

Moreover, $L(R^N)$ is the set of all bounded linear mappings from R^N to R^N.

If x_k in Eq.(2) is an approximate solution of Eq.(3), obtained by using an iterative method to solve the linear equations, then we refer to the resulting combined method as a Newton-like-iterative method. Such methods have received much attention in Refs.[1-4]. In Ref.[5], T.J. Ypma introduced this method and studied its convergence results, which generalized those of Refs.[2-4]. His approach is to regard Newton-like-iterative methods as inexact Newton methods[6-7], as suggested in Ref.[6]. The relevant theory is applied to produce conditions for convergence, as well as radius of convergence and rate of convergence results.

Since inexact Newton methods were proposed, there have been plentiful results concerning its local convergence[6,8-10], semi-local convergence[11-16] and global convergence[17-18], as well as its generalizations and applications to different numerical fields[19-20]. Except for convergence results for inexact Newton methods, Morini[9] and Jinhai Chen-Weiguo Li[8] considered respectively inexact Newton-like methods and their local convergence theorems under different assumptions as well. To mention the results, suppose x_0 is a given initial guess, and then the iterative form is as follows:

For $k=0$ step 1 until convergence do

Find some step s_k which satisfies

$$A_k s_k = -f(x_k) + r_k \quad \text{where } \frac{\| P_k r_k \|}{\| P_k f(x_k) \|} \leqslant \eta_k, \tag{4}$$

Set $x_{k+1} = x_k + s_k$;

where η_k is a sequence of forcing terms and P_k is an invertible matrix for each k. It is worth noting that residuals of this form are used in iterative Newton methods if preconditioning is applied, and that P_k changes with index k if A_k does. Taking account of the affine invariance of inexact Newton-like methods, here we choose $P_k = A_k^{-1}$ especially. So the condition is

$$\frac{\| A_k^{-1} r_k \|}{\| A_k^{-1} f(x_k) \|} \leqslant \eta_k. \tag{5}$$

Apparently, the process is inexact Newton method if $A_k = f'(x_k)$ and inexact modified Newton method if $A_k = f'(x_0)$.

In this paper, we give different theorems for Newton-like-iterative methods by applying the theory of inexact Newton-like methods[8-9], which are simpler and easier to apply. We first summarize, in Section 2, the convergence results for inexact Newton-like methods given in Ref.[8] and provide a new theorem as well. In Section 3, we show that Newton-like-iterative methods are themselves inexact Newton-like methods, and then derive the corresponding convergence results in Section 4.

In what follows, I denotes the identity operator. $\|x\|$ for $x \in R^N$ denotes some vector norm of x, and $\|A\|$ for $A\in L(R^N)$ denotes the subordinate matrix norm of A. The notion $S(x,r)=\{y\in R^N \mid \|x-y\|<r\}$ denotes the open ball with center x and radius r.

2 Inexact Newton-like methods

Firstly, we conclude the following convergence theorem for inexact Newton-like method from Ref. [8], which produces conditions for its local convergence as well as the radius of the convergence ball.

Theorem 2.1[8] Suppose x^* is the only solution of Eq. (1) in $S(x^*,r)$. Assume that f has a continuous derivative in $S(x^*,r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the following Lipschitz condition:

$$\|f'(x^*)^{-1}(f'(y)-f'(x))\|\leqslant\lambda\|y-x\|,\quad \forall\, x,y\in S(x^*,r). \tag{6}$$

Suppose further $A(x)$ is an approximation to the Jacobian $f'(x)$ for $x\in S(x^*,r)$, $A(x)$ is invertible and such that

$$\|A(x)^{-1}f'(x)\|\leqslant\omega_1,\quad \|A(x)^{-1}f'(x)-I\|\leqslant\omega_2. \tag{7}$$

Let $\nu_k\leqslant\nu<1$ and $0<r<1/\lambda$ satisfy

$$\frac{(1+\nu)\lambda\omega_1 r}{2(1-\lambda r)}+\omega_2+\nu\omega_1\leqslant 1. \tag{8}$$

Then the inexact Newton-like method $\{x_k\}$ is convergent for all $x_0\in S(x^*,r)$ and satisfies $\|x_{k+1}-x^*\|\leqslant q\|x_k-x^*\|$, $k=0,1,\cdots$, where $q\in(0,1)$ is given by

$$q=\frac{(1+\nu)\lambda\omega_1\|x_0-x^*\|}{2(1-\lambda\|x_0-x^*\|)}+\omega_2+\nu\omega_1.$$

Taking $A(x)=f'(x)$, i.e., $\omega_1=1$, $\omega_2=0$ in Theorem 2.1, we obtain the convergent results for inexact Newton method proposed in Ref. [5]. Clearly, if the equal sign holds in Eq. (8), we get the optimal radius of convergence ball for inexact Newton methods, with $r=2(1-\nu)/\lambda(3-\nu)$.

Furthermore, let $\nu=0$, then it merges into Newton's method and the estimate for the radius of convergence ball is known to be sharp[21].

In the following, we give a new convergence theorem for inexact Newton-like methods under different assumptions.

Theorem 2.2 Suppose x^* is the only solution of Eq. (1) in $S(x^*,r)$. Assume that f has a continuous derivative in $S(x^*,r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the following Lipschitz condition:

$$\|f'(x^*)^{-1}f'(x)-I\|\leqslant\lambda\|x-x^*\|,\quad \forall\, x\in S(x^*,r). \tag{9}$$

Suppose further $A(x)$ is an approximation to the Jacobian $f'(x)$ for $x\in S(x^*,r)$, $A(x)$ is invertible and such that

$$\|A(x)^{-1}f'(x^*)\|\leqslant\omega_1,\quad \|A(x)^{-1}f'(x^*)-I\|\leqslant\omega_2. \tag{10}$$

Let $\nu_k\leqslant\nu<1$ and $0<r<1/\lambda$ satisfy

$$\frac{(1+\nu)\lambda\omega_1 r}{2}+\omega_2+\nu\omega_1\leqslant 1. \tag{11}$$

Then the inexact Newton-like method $\{x_k\}$ is convergent for all $x_0\in S(x^*,r)$ and satisfies $\|x_{k+1}-x^*\|\leqslant q\|x_k-x^*\|$, $k=0,1,\cdots$, where $q\in(0,1)$ is given by

$$q=\frac{(1+\nu)\lambda\omega_1\|x_0-x^*\|}{2}+\omega_2+\nu\omega_1.$$

Proof Arbitrarily choosing $x_0\in S(x^*,r)$, where r is determined by Eq. (11), $q<1$ follows directly.

Since $r<1/\lambda$, by Eq.(9) and Numann Lemma, we have that $f'(x)^{-1}$ exists for all $x_0\in S(x^*,r)$ as well as

$$\|f'(x)^{-1}f'(x^*)\|\leqslant\frac{1}{1-\lambda\|x-x^*\|}.$$

Moreover, it is easy to get that

$$\|f'(x^*)^{-1}(f(x_k)-f(x^*)-f'(x^*)(x_k-x^*))\|\leqslant\frac{\lambda}{2}\|x_k-x^*\|^2.$$

Now if $x_k\in S(x^*,r)$, we have by the definition of inexact Newton-like methods Eq.(4) that

$$\begin{aligned}x_{k+1}-x^* &= x_k-x^*-A_k^{-1}f(x_k)+A_k^{-1}r_k\\ &=-A_k^{-1}(f(x_k)-f(x^*)-f'(x^*)(x_k-x^*))-A_k^{-1}(f'(x^*)-A_k)(x_k-x^*)+A_k^{-1}r_k.\end{aligned}$$

Using Eqs.(9), (10) and (5), we obtain

$$\begin{aligned}\|x_{k+1}-x^*\|\leqslant&\|A_k^{-1}f'(x^*)\|\cdot\|f'(x^*)^{-1}(f(x_k)-f(x^*)-f'(x^*)(x_k-x^*))\|+\\ &\|A_k^{-1}(f'(x^*)-A_k)\|\cdot\|x_k-x^*\|+\|A_k^{-1}r_k\|\leqslant\\ &\frac{\lambda\omega_1}{2}\|x_k-x^*\|^2+\omega_2\|x_k-x^*\|+\nu_k\|A_k^{-1}f(x_k)\|.\end{aligned}$$

In what follows, we give estimation on $\|A_k^{-1}f(x_k)\|$

$$\begin{aligned}\|A_k^{-1}f(x_k)\|=&\|A_k^{-1}(f(x_k)-f(x^*)-f'(x^*)(x_k-x^*)+f'(x^*)(x_k-x^*))\|\leqslant\\ &\|A_k^{-1}f'(x^*)\|\cdot\|f'(x^*)^{-1}(f(x_k)-f(x^*)-f'(x^*)(x_k-x^*))\|+\\ &\|A_k^{-1}f'(x^*)\|\cdot\|x_k-x^*\|\leqslant\frac{\lambda\omega_1}{2}\|x_k-x^*\|^2+\omega\|x_k-x^*\|.\end{aligned}$$

Therefore, by the above calculation, we get

$$\|x_{k+1}-x^*\|\leqslant\left(\frac{(1+\nu_k)\lambda\omega_1}{2}\|x_k-x^*\|+\nu_k\omega_1+\omega_2\right)\|x_k-x^*\|.$$

Taking $k=0$ above, we obtain $\|x_1-x^*\|\leqslant q\|x_0-x^*\|<\|x_0-x^*\|$. Hence, $x_1\in S(x^*,r)$. This shows that Eq.(4) can be continued an infinite number of times. By mathematical induction, all x_k belong to $S(x^*,r)$ and $\|x_k-x^*\|$ decreases monotonically. Therefore, remembering $\nu_k\leqslant\nu<1$, for all $k\geqslant 0$, we have

$$\|x_{k+1}-x^*\|\leqslant\left(\frac{(1+\nu)\lambda\omega_1}{2}\|x_0-x^*\|+\nu\omega_1+\omega_2\right)\|x_k-x^*\|=q\|x_k-x^*\|.$$

The proof is completed.

3 Newton-like-iterative methods

In this section, we shall show that Newton-like-iterative methods are inexact Newton-like methods. Local convergence results follow from the above theorems.

Splitting is one of the basic principles used to generate iterative methods for solving a system of linear equations. Applying the technique to Eq.(3), and decomposing A_k as

$$A_k=B_k-C_k,\quad B_k,C_k\in L(R^N).$$

the resulting Newton-like-iterative method takes the form

$$x_{k+1}=x_k+s_k^{M_k}\quad k=0,1,\cdots,$$

where B_k is nonsingular and the system of linear equations $B_kx=d$, $d\in R^N$, is in some sense easy to solve. Here M_k is some positive integer and $s_k^{M_k}$ is calculated by solving successively the system of linear equations

$$B_ks_k^{j+1}=C_ks_k^j-f(x_k)\quad j=0,1,\cdots,M_k-1\tag{12}$$

with $s_k^0 \in R^N$ some selected starting value. Defining $H_k \equiv B_k^{-1} C_k$, by a simple analysis, we have Eq.(13)[2]

$$x_{k+1} = x_k - (I - H_k^{M_k}) A_k^{-1} f(x_k) + H_k^{M_k} s_k^0. \tag{13}$$

It is worthy to notice that a sufficient condition for the convergence of the sequence $\{s_k^j\}$ generated by Eq.(12) to the solution $-A_k^{-1} f(x_k)$ of Eq.(3) is that $\|H_k\| \leqslant 1$. Comparing Eq.(13) with Eq.(4), it is clear that Newton-like-iterative methods are inexact Newton-like methods.

If we choose $B_k \equiv A_k$, then Eq.(13) reduces to

$$x_{k+1} = x_k - A_k^{-1} f(x_k)$$

since $H_k = 0$. The analysis of Eq.(13) therefore generalizes the analysis of the standard Newton-like methods in Eqs.(2) and (3).

4 Convergence theorems and corollaries

Suppose the Newton-like-iterative methods are defined in Section 3. Our main result in this section is as follows.

Theorem 4.1 Suppose x^* is the only solution of (1) in $S(x^*, r)$, where r satisfies Eq.(8). Assume that f has a continuous derivative in $S(x^*, r)$, $f'(x^*)^{-1}$ exists and Eq.(6) holds. Let $A(x)$ be an approximation to the Jacobian $f'(x)$ for $x \in S(x^*, r)$. $A(x)$ is invertible and Eq.(7) is satisfied. Suppose

$$\gamma_k \equiv \|H_k^{M_k}\| \frac{\|A_k^{-1} f(x_k) + s_k^0\|}{\|A_k^{-1} f(x_k)\|} \leqslant \nu_k \leqslant \nu < 1. \tag{14}$$

Then the Newton-like-iterative method $\{x_k\}$ is convergent for all $x_0 \in S(x^*, r)$ and satisfies $\|x_{k+1} - x^*\| \leqslant q \|x_k - x^*\|$, $k = 0, 1, \cdots$, where $q < 1$ is given in Theorem 2.1.

Proof By the definition of the inexact Newton-like methods, it follows that

$$A_k^{-1} r_k = s_k + A_k^{-1} f(x_k).$$

Then since Eq.(13), we get

$$S_k^{M_k} + A_k^{-1} f(x_k) = -(I - H_k^{M_k}) A_k^{-1} f(x_k) + H_k^{M_k} s_k^0 + A_k^{-1} f(x_k) = H_k^{M_k} (A_k^{-1} f(x_k) + s_k^0).$$

Thus, $\dfrac{\|A_k^{-1} r_k\|}{\|A_k^{-1} f(x_k)\|} \equiv \dfrac{\|H_k^{M_k}(A_k^{-1} f(x_k) + s_k^0)\|}{\|A_k^{-1} f(x_k)\|} \leqslant \|H_k^{M_k}\| \dfrac{\|A_k^{-1} f(x_k) + s_k^0\|}{\|A_k^{-1} f(x_k)\|} = \gamma_k$.

The results then follow from Theorem 2.1.

Comparing with the relative result proposed in Ref.[5], the above theorem produce simpler and clearer conditions for forcing sequence of Newton-like-iterative methods. Here, in Ref.[5], γ_k is formulated by

$$\gamma_k = (1 + \|H_k^{M_k}\|) \omega_2 + \|H_k^{M_k}\| \frac{\|A_k^{-1} f(x_k) + s_k^0\|}{\|A_k^{-1} f(x_k)\|}.$$

Similarly, we get the following theorem from Theorem 2.2, which seems not appeared in the literature.

Theorem 4.2 Suppose x^* is the only solution of Eq.(1) in $S(x^*, r)$, where r satisfies Eq.(11). Assume that f has a continuous derivative in $S(x^*, r)$, $f'(x^*)^{-1}$ exists and Eq.(9) holds. Let $A(x)$ be an approximation to the Jacobian $f'(x)$ for $x \in S(x^*, r)$. $A(x)$ is invertible and Eq.(10) is satisfied. Suppose further (14) holds. Then the Newton-like-iterative method $\{x_k\}$ is convergent for all $x_0 \in S(x^*, r)$ and satisfies $\|x_{k+1} - x^*\| \leqslant q \|x_k - x^*\|$, $k = 0, 1, \cdots$, where $q < 1$ is given in Theorem 2.2.

By simple deduction, we obtain the following corollary

Corollary 4.3 Let the assumptions of Theorem 4.1 hold, up to Newton-like-iterative methods, with $s_k^0=0$, $k=0,1,\cdots$. Suppose that

$$\| H_k^{M_k} \| \leqslant \nu_k \leqslant \nu < 1 \quad k=0,1,\cdots$$

for some sequence $\nu_k \subseteq [0,1)$. Then all the consequences of Theorem 4.1 hold. If in addition $A_k \equiv f'(x_k)$, $\| H_k \| \leqslant \eta < 1$ and $\lim_{k\to+\infty} M_k = +\infty$, then the convergence is Q-superlinear.

Remark Taking $A_k \equiv f'(x_k)$ in Eq.(4), the inexact Newton-like methods merge into inexact Newton methods. The rate of convergence follows from the classical results of inexact Newton methods [6,10].

References:

[1] BANK R E, ROSE D J. Global approximate Newton methods[J]. Numer Math, 1981(37): 279-295.

[2] ORTEGA J M, RHEINBOLDT W C. Iterative Solution of Nonlinear Equations in Several Variables[M]. New York: Academic Press, 1970.

[3] RHEINBOLDT W C. Methods for Solving Systems of Nonlinear Equations[M]. Philadephia: SIAM, 1974.

[4] SHERMAN A H. On Newton-iterative methods for the solution of systems of nonlinear equations[J]. SINUM, 1978 (15): 755-771.

[5] YPMA T J. Convergence of Newton-like-iterative methods[J]. Numer Math, 1984(45): 241-251.

[6] DEMBO R S, EISENSTAT S C, STEIHAUG T. Inexact Newton methods[J]. SIAM J Numer Anal, 1982(19): 400-408.

[7] DENNIS J E, WALKER H F. Local convergence theorems for quasi-Newton methods[R].[2009-11-06].http://hdl.handle.net/1813/7498. Cornell Computer Science, 1979.

[8] CHEN Jinhai, LI Weiguo. Convergence behaviour of inexact Newton methods under weak Lipschitz condition[J]. J Comput Appl Math, 2006(191): 143-164.

[9] MORINI B. Convergence behaviour of inexact Newton method[J]. Math Comp, 1999(68): 1605-1613.

[10] YPMA T J. Local convergence of inexact Newton methods[J]. SIAM J Numer Anal, 1984(21): 583-590.

[11] ARGROS I K. A new convergence theorem for the inexact Newton methods based on assumptions involving the second Fréchet derivative[J]. Comp Math Appl, 1999(37): 109-115.

[12] BAI Zhongzhi, TONG Peili. On the affine invariant convergence theorems of inexact Newton method and Broyden's method[J]. Journal of UEST of China, 1994(23): 535-539.

[13] GUO Xueping. On semilcal convergence of inexact Newton methods[J]. Journal of Computational Mathematics, 2007 (25): 231-242.

[14] MORET I. A Kantorovich type theorem for inexact Newton methods[J]. Numer Funct Anal Optim, 1989(10): 351-365.

[15] WU Min. A new semi-local convergence theorem for the inexact Newton methods[J]. Appl Math Comput, 2008 (200): 80-86.

[16] ZHOU Jiali, ZHANG Shuyou, YANG Guoping. A convergence theorem for the inexact Newton methods based on Hölder continuous Fréchet derivative[J]. Appl Math Comput, 2008(197): 206-211.

[17] EISENSTAT S C, WALKER H F. Globally convergent inexact Newton methods[J]. SIAM J Optimization, 1994 (4): 393-422.

[18] SOLODOV M V, SVAITER B F. Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods [M]. London: Kluwer Academic Publishers, 1998.

[19] JACKSON K R. The numerical solution of large systems of stiff IVPs for ODEs[J]. Appl Numer Math, 1996(20): 5-20.

[20] MARTINEZ J M, QI L. Inexact Newton methods for solving nonsmooth equations[J]. J Comput Appl Math, 1995 (60): 127-145.

[21] WANG Xinghua. Convergence of Newton's method and uniqueness of the solution of equations in Banach space[J]. IMA J Numer Anal, 2000(20): 123-134.