

# Comparison of $It\hat{o}$ and $Stratonovich$ integrals

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**Abstract:** We introduce the definitions of the  $It\hat{o}$  integral and the  $Stratonovich$  integral and present the  $It\hat{o}$  formula used to calculate the  $It\hat{o}$  integral. Also, the equation, which represents the relationship between the  $It\hat{o}$  integral and the  $Stratonovich$  integral, is derived. Through examples, we solve the stochastic differential equations by applying the two kinds of integrals respectively and compare the solutions obtained from the two different methods. At the end, we investigate the advantages and disadvantages of these two kinds of integrals in different practical applications.

**Key words:**  $It\hat{o}$  integral;  $Stratonovich$  integral;  $It\hat{o}$  formula; stochastic differential equation

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## $It\hat{o}$ 积分和 $Stratonovich$ 积分的比较

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**摘 要:** 引入了  $It\hat{o}$  积分和  $Stratonovich$  积分的定义,介绍了计算  $It\hat{o}$  积分的  $It\hat{o}$  公式,并讨论了  $It\hat{o}$  积分和  $Stratonovich$  积分之间的关联公式。通过实例,运用这两种积分分别求解随机微分方程,并将结果进行了对比。最后,对它们在不同的实际应用中各自具有的优、缺点进行了讨论。

**关键词:**  $It\hat{o}$  积分;  $Stratonovich$  积分;  $It\hat{o}$  公式; 随机微分方程

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### 1 Introduction

In this paper, we consider the two kinds of stochastic integrals, the  $It\hat{o}$  integral and the  $Stratonovich$  integral. Let  $(\Omega, \mathcal{F})$  be a measure space with the probability measure  $P$  and  $B_t(\omega)$  be a  $n$ -dimensional Brownian motion. Assume that  $\mathcal{F}_t = \mathcal{F}_t^{(n)}$  is the  $\sigma$ -algebra generated by the random variables

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$\{B_i(s)\}_{1 \leq i \leq n, 0 \leq s \leq t}$ . We denote by  $V(S, T)$  the class of functions.

$$f(t, \omega): [0, \infty) \times \Omega \rightarrow \mathbb{R}$$

such that

- 1)  $(t, \omega) \rightarrow f(t, \omega)$  is  $B \times \mathcal{F}$ -measurable, where  $B$  denotes the Borel  $\sigma$ -algebra on  $[0, \infty)$ ;
- 2)  $f(t, \omega)$  is  $\mathcal{F}_t$ -adapted;
- 3)  $E\left[\int_0^T f(t, \omega)^2 dt\right] < \infty$ .

We adopt  $L^2(P)$  to be a *Hilbert* space which is a complete inner product space with the following inner product.

$$(X, Y)_{L^2(P)} = E[X \cdot Y]; X, Y \in L^2(P).$$

**Definition 1** (*Itô* integral) Suppose  $f \in V(0, T)$  and that  $t \rightarrow f(t, \omega)$  is continuous for a. a.  $\omega$ . Then the *Itô* integral is defined by

$$\int_0^T f(t, \omega) dB_t(\omega) = \lim_{\Delta t_j \rightarrow 0} \sum_j f(t_j, \omega) \Delta B_j \text{ in } L^2(P).$$

**Definition 2** (*Stratonovich* integral) Suppose  $f \in V(0, T)$  and that  $t \rightarrow f(t, \omega)$  is continuous for a. a.  $\omega$ . Then the *Stratonovich* integral of  $f$  is defined by

$$\int_0^T f(t, \omega) \circ dB_t(\omega) = \lim_{\Delta t_j \rightarrow 0} \sum_j f(t_j^*, \omega) \Delta B_j, \text{ where } t_j^* = \frac{1}{2}(t_j + t_{j+1}),$$

whenever the limit exists in  $L^2(P)$ .

## 2 Itô formula

**Theorem 1** (*Itô* formula) Let  $X_t$  be an *Itô* process given by

$$dX_t = udt + vdB_t.$$

Assume  $g(t, x) \in C^2([0, \infty) \times R)$  (i. e.  $g$  is twice continuously differentiable on  $[0, \infty) \times R$ ).

Then

$$Y_t = g(t, X_t)$$

is also an *Itô* process, and we have

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \cdot \frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2, \quad (1)$$

where  $(dX_t)^2 = (dX_t) \cdot (dX_t)$  is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, dB_t \cdot dB_t = dt.$$

**Proof** The argument is standard. We only give the simple lines for completeness.

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \left[ \frac{\partial^2 g}{\partial t^2}(t, X_t)(dt)^2 + 2 \cdot \frac{\partial^2 g}{\partial t \cdot \partial x}(t, X_t)dX_t \cdot dt + \frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2 \right].$$

Using the *Itô* formula, we can compute some *Itô* integrals.

**Example 1** Assuming  $B_0 = 0$ , we calculate the *Itô* integral  $\int_0^t B_s dB_s$ .

**Solution** It is very complicated to calculate the *Itô* integral by the definition directly. However, it is much easier by using the *Itô* formula. So let  $X_t = B_t$ , and  $g(t, x) = \frac{1}{2}x^2$ . Then we have

$$Y_t = g(t, X_t) = \frac{1}{2}B_t^2.$$

By the *Itô* formula (1), we get

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dB_t + \frac{1}{2} \cdot \frac{\partial^2 g}{\partial x^2} (dB_t)^2 = B_t dB_t + \frac{1}{2} (dB_t)^2 = B_t dB_t + \frac{1}{2} dt.$$

Then we obtain

$$d\left(\frac{1}{2} B_t^2\right) = B_t dB_t + \frac{1}{2} dt, \text{ or } \frac{1}{2} B_t^2 = \int_0^t B_s dB_s + \frac{1}{2} t.$$

So, we get the value of this *Itô* integral as the following

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t.$$

### 3 Relationship between the *Itô* integral and the *Stratonovich* integral

**Theorem 2** Suppose  $f \in V(0, T)$  and that  $t \rightarrow f(t, \omega)$  is continuous for a. a.  $\omega$ . Then

$$\int_0^T f(t, \omega) \circ dB_t = \int_0^T f(t, \omega) dB_t + \frac{1}{2} \int_0^T \frac{\partial f(t, \omega)}{\partial B_t} dt. \quad (2)$$

**Proof** Suppose  $f \in V(0, T)$  and that  $t \rightarrow f(t, \omega)$  is continuous for a. a.  $\omega$ . Then,

$$\begin{aligned} \int_0^T f(t, \omega) \circ dB_t(\omega) &= \lim_{\Delta_j \rightarrow 0} \sum_j f(t_j^*, \omega) \Delta B_j = \lim_{\Delta_j \rightarrow 0} \sum_j f(t_j^*, \omega) (B_{t_{j+1}} - B_{t_j}) = \\ &= \lim_{\Delta_j \rightarrow 0} \left\{ \sum_j f(t_j^*, \omega) (B_{t_{j+1}} - B_{t_j^*}) + \sum_j f(t_j, \omega) (B_{t_j^*} - B_{t_j}) + \sum_j [f(t_j^*, \omega) - f(t_j, \omega)] (B_{t_j^*} - B_{t_j}) \right\} = \\ &= \lim_{\Delta_j \rightarrow 0} \left\{ \sum_j f(t_j^*, \omega) (B_{t_{j+1}} - B_{t_j^*}) + \sum_j f(t_j, \omega) (B_{t_j^*} - B_{t_j}) \right\} + \lim_{\Delta_j \rightarrow 0} \sum_j [f(t_j^*, \omega) - f(t_j, \omega)] (B_{t_j^*} - B_{t_j}). \end{aligned}$$

It is clear that the first limit on the left side of the last equality is  $\int_0^T f(t, \omega) dB_t$ . For the second limit of the last equality, we see from the Talor's formula, there is a  $\xi_j \in [B_{t_j}, B_{t_j^*}]$  such that

$$\begin{aligned} \lim_{\Delta_j \rightarrow 0} \sum_j [f(t_j^*, \omega) - f(t_j, \omega)] (B_{t_j^*} - B_{t_j}) &= \lim_{\Delta_j \rightarrow 0} \sum_j \left[ \frac{\partial f(\xi_j, \omega)}{\partial B_t} (B_{t_j^*} - B_{t_j}) \right] (B_{t_j^*} - B_{t_j}) = \\ &= \lim_{\Delta_j \rightarrow 0} \sum_j \frac{\partial f(\xi_j, \omega)}{\partial B_t} (B_{t_j^*} - B_{t_j})^2 = \lim_{\Delta_j \rightarrow 0} \frac{1}{2} \sum_j \frac{\partial f(\xi_j, \omega)}{\partial B_t} (B_{t_{j+1}} - B_{t_j})^2 = \frac{1}{2} \lim_{\Delta_j \rightarrow 0} \sum_j \frac{\partial f(\xi_j, \omega)}{\partial B_t} (\Delta B_{t_j})^2. \end{aligned}$$

By  $\sum_j (\Delta B_{t_j})^2 \rightarrow T$  in  $L^2(P)$  as  $\Delta_{t_j} \rightarrow 0$ , we find that

$$\lim_{\Delta_j \rightarrow 0} \sum_j [f(t_j^*, \omega) - f(t_j, \omega)] (B_{t_j^*} - B_{t_j}) = \frac{1}{2} \int_0^T \frac{\partial f(t, \omega)}{\partial B_t} dt.$$

Now we can use the **Theorem 2** to compute some *Stratonovich* integrals.

**Example 2** Assuming  $B_0 = 0$ , we calculate the *Stratonovich* integral  $\int_0^t B_s \circ dB_s$ .

**Solution** By the formula (2), we know

$$\int_0^t f(s, \omega) \circ dB_s = \int_0^t f(s, \omega) dB_s + \frac{1}{2} \int_0^t \frac{\partial f(s, \omega)}{\partial B_s} ds, \text{ where } f(s, x) = B_s.$$

Then, we have

$$\begin{aligned} \int_0^t f(s, \omega) \circ dB_s &= \int_0^t f(s, \omega) dB_s + \frac{1}{2} \int_0^t \frac{\partial f(s, \omega)}{\partial B_s} dt = \int_0^t B_s dB_s + \frac{1}{2} \int_0^t 1 \cdot dt = \\ &= \int_0^t B_s dB_s + \frac{1}{2} t = \frac{1}{2} B_t^2 - \frac{1}{2} t + \frac{1}{2} t = \frac{1}{2} B_t^2. \end{aligned}$$

We see the different values of the two kinds of integrals clearly through the Example 1 and the Example 2.

#### 4 Application in the stochastic differential equations

Example 3 Solve the following stochastic equation, which is a well-known population growth model

$$dN_t = rN_t dt + \alpha N_t dB_t. \quad (3)$$

**Solution** The equation (3) can be written as

$$\frac{dN_t}{N_t} = rdt + \alpha dB_t, \text{ or } \int_0^t \frac{dN_s}{N_s} = rt + \alpha B_t (B_0 = 0).$$

By the *Itô* formula, we have

$$d(\ln N_t) = \frac{1}{N_t} dN_t + \frac{1}{2} \left( -\frac{1}{N_t^2} \right) (dN_t)^2.$$

By the equation (3), we obtain  $(dN_t)^2 = (rN_t dt + \alpha N_t dB_t)^2 = \alpha^2 N_t^2 (dB_t)^2 = \alpha^2 N_t^2 dt$ .

So we get

$$d(\ln N_t) = \frac{dN_t}{N_t} - \frac{1}{2N_t^2} \alpha^2 N_t^2 dt = \frac{dN_t}{N_t} - \frac{1}{2} \alpha^2 dt.$$

Hence,

$$\int_0^t \frac{dN_s}{N_s} = \int_0^t d(\ln N_s) = \int_0^t \frac{dN_s}{N_s} - \int_0^t \frac{1}{2} \alpha^2 ds.$$

Then we can conclude

$$\ln \frac{N_t}{N_0} = rt + \alpha B_t - \frac{1}{2} \alpha^2 t = \left( r - \frac{1}{2} \alpha^2 \right) t + \alpha B_t.$$

The solution is

$$N_t = N_0 \exp \left( \left( r - \frac{1}{2} \alpha^2 \right) t + \alpha B_t \right). \quad (4)$$

Example 4 The *Stratonovich* interpretation of stochastic equation (3) is

$$d\bar{N}_t = r\bar{N}_t dt + \alpha \bar{N}_t \circ dB_t.$$

Solve this stochastic equation.

**Solution** By the **Theorem 2**, we have

$$d\bar{N}_t = r\bar{N}_t dt + \alpha \bar{N}_t dB_t + \frac{1}{2} \alpha \left( \frac{\partial \bar{N}_t}{\partial B_t} \right) dt = r\bar{N}_t dt + \alpha \bar{N}_t dB_t + \frac{1}{2} \alpha^2 \bar{N}_t dt = \left( r + \frac{1}{2} \alpha^2 \right) \bar{N}_t dt + \alpha \bar{N}_t dB_t,$$

where  $\frac{\partial \bar{N}_t}{\partial B_t} = \alpha \bar{N}_t$ .

We can regard “ $r + \frac{1}{2} \alpha^2$ ” in this equation as the “ $r$ ” in the equation (3). By using the solution showed with the formula (4), we have

$$\bar{N}_t = N_0 \exp \left( \left( r + \frac{1}{2} \alpha^2 - \frac{1}{2} \alpha^2 \right) t + \alpha B_t \right) = N_0 \exp (rt + \alpha B_t).$$

The solutions  $N_t$  and  $\bar{N}_t$  are both of the form

$$X_t = X_0 \exp (\mu t + \alpha B_t) \quad (\text{Here } \mu, \alpha \text{ are both constants}).$$

We call such a process Geometric Brownian motion. It is also an important model for stochastic prices in economics<sup>[1]</sup>.

#### 5 Contrast between the *Itô* integral and the *Stratonovich* integral

At the end, let us return to the population growth model in the Example 3. We know that  $N_t$  is a solution of the stochastic equation (3), and

$$N_t = N_0 + \int_0^t r N_s ds + \int_0^t \alpha N_s dB_s. \quad (5)$$

For some suitable interpretation of the last integral in the equation (5), the  $It\hat{o}$  interpretation of an integral is just one of the several reasonable choices. However, the  $Stratonovich$  integral is another choice, usually leading to a different result. So the question is: Which interpretation of the last integral in the equation (5) makes the equation the “exact” mathematical model for this equation? The  $Stratonovich$  interpretation in some situations may be the most appropriate. Choose  $t$ -continuously differentiable processes  $B_t^{(n)}$  such that for a. a.  $\omega$ ,

$$B_t^{(n)}(t, \omega) \rightarrow B(t, \omega), n \rightarrow \infty$$

uniformly (in  $t$ ) in bounded intervals. For each  $\omega$  let  $N_t^{(n)}(\omega)$  be the solution of the corresponding (deterministic) differential equation

$$\frac{dN_t}{dt} = b(t, N_t) + \sigma(t, N_t) \frac{dB_t^{(n)}}{dt}.$$

Then, for a. a.  $\omega$ ,

$$N_t^{(n)}(t, \omega) \rightarrow N(t, \omega), n \rightarrow \infty$$

uniformly (in  $t$ ) in bounded intervals.

It turns out<sup>[2-3]</sup> that this solution  $N_t$  coincides with the solution of the equation (5) obtained by using the  $Stratonovich$  integral

$$N_t = N_0 + \int_0^t b(s, N_s) ds + \int_0^t \sigma(s, N_s) \circ dB_s. \quad (6)$$

This outcome implies that  $N_t$  is the solution of the following modified the  $It\hat{o}$  equation,

$$N_t = N_0 + \int_0^t b(s, N_s) ds + \frac{1}{2} \int_0^t \sigma'(s, N_s) \sigma(s, N_s) ds + \int_0^t \sigma(s, N_s) dB_s, \quad (7)$$

where  $\sigma'$  denotes the derivative of  $\sigma(t, x)$  w. r. t.  $x$ <sup>[4]</sup>.

Therefore, from this point of view it seems reasonable to use the  $Stratonovich$  interpretation of the equation (6), and not the  $It\hat{o}$  interpretation of the equation (5) as the model for the original white noise equation. However, the specific feature of the  $It\hat{o}$  model of “not looking into the future”<sup>[5]</sup> seems to be a reason for choosing the  $It\hat{o}$  interpretation in many cases, for example in biology<sup>[6]</sup>. Note that equation (5) and (7) coincide if  $\sigma(t, x)$  does not depend on  $x$ <sup>[7]</sup>.

By the **Theorem 2**, we can find that there is no second order term in the  $Stratonovich$  analogue of the  $It\hat{o}$  transformation formula. It can be said that the  $Stratonovich$  integral has the advantage of leading to ordinary chain rule formulas under a transformation. This advantage makes the  $Stratonovich$  integral good to use for example in connection with stochastic differential equations on manifolds<sup>[8-9]</sup>. However, the  $Stratonovich$  integrals are not martingales, but the  $It\hat{o}$  integrals are. This gives the  $It\hat{o}$  integral an important computational advantage, even though it does not behave so nicely under transformations.

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